

induces a map

$$P_e G \longrightarrow L^2([0, 1], \mathfrak{g}), \quad g \mapsto g^{-1} * 0 = g^{-1} g',$$

which turns out to be an isometry. Composed with the endpoint map $P_e G \rightarrow G, g \mapsto g(1)$ its inverse yields the desired Riemannian submersion

$$\begin{aligned} \pi : L^2([0, 1], \mathfrak{g}) &\longrightarrow P_e(G) \longrightarrow G \\ u = g^{-1} * 0 &\longmapsto g \longmapsto g(1). \end{aligned}$$

From this description we see that the fibers of π are precisely the orbits of ΩG under the above action. Denoting the associated horizontal and vertical distributions on $L^2([0, 1], \mathfrak{g})$ by \mathcal{H} and \mathcal{V} , we see that

$$\begin{aligned} \mathcal{V}_0 &= \left\{ u \in L^2([0, 1], \mathfrak{g}) \mid \int_0^1 u \, dt = 0 \right\} \quad \text{and thus} \\ \mathcal{H}_0 &= \{ u \in L^2([0, 1], \mathfrak{g}) \mid u \equiv \text{const} \}. \end{aligned}$$

We will therefore frequently not distinguish between \mathcal{H}_0 and \mathfrak{g} . The projection $L^2([0, 1], \mathfrak{g}) \rightarrow \mathcal{H}_0$ is given by $u^{\mathcal{H}_0} = \int_0^1 u \, dt$. Also, a simple calculation shows that the derivative of the isometry $P_e G \rightarrow L^2([0, 1], \mathfrak{g})$ yields an identification of $T_e \Omega G = \{ u \in H^1([0, 1], \mathfrak{g}) \mid u(0) = u(1) = 0 \}$ and \mathcal{V}_0 given by $u \mapsto u'$.

The isometries induced by $H^1([0, 1], G)$ preserve the fibration associated to π ; more precisely, we have the relation

$$\pi(g * u) = g(0) \pi(u) g^{-1}(1),$$

for $g \in H^1([0, 1], G)$. Conversely, this also shows that isometries of G can be lifted to $L^2([0, 1], \mathfrak{g})$: if $\varphi : G \rightarrow G$ is given by $\varphi(x) = g_0 x g_1^{-1}$ and g any curve joining g_0 and g_1 then the isometry of $L^2([0, 1], \mathfrak{g})$ induced by g covers φ :

$$\begin{array}{ccc} L^2([0, 1], \mathfrak{g}) & \xrightarrow{g^*} & L^2([0, 1], \mathfrak{g}) \\ \pi \downarrow & & \downarrow \pi \\ G & \xrightarrow{\varphi} & G. \end{array}$$

As in finite dimensions there is the usual relation between skew adjoint operators and isometries on a Hilbert space given by the exponential map, which also in this context is a local diffeomorphism near